NOTES ON THE THEORIES OF JUPITER AND SATURN.

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THE coordinates usually preferred by astronomers are the logarithm of the radius vector, the longitude and the latitude. We suppose that the two last are referred to the plane of maximum areas. Let these coordinates be denoted by the symbols $\log \rho$, λ and β; and let the subscript $(_0)$ be applied to λ and β when we wish to designate the similar coordinates corresponding to the variables x, y, z, x', y', z'. Then we have

$$\begin{aligned}
\rho \cos\beta\cos\lambda &= r\cos\beta_0\cos\lambda_0 + \varkappa r'\cos\beta'_0\cos\lambda'_0,\\
\rho \cos\beta\sin\lambda &= r\cos\beta_0\sin\lambda_0 + \varkappa r'\cos\beta'_0\sin\lambda'_0,\\
\rho \sin\beta &= r\sin\beta_0 + \varkappa r'\sin\beta'_0
\end{aligned}$$

From the first two equations are readily obtained the following two:—

$$\begin{aligned}
\rho\cos\beta\cos(\lambda-\lambda_0) &= r\cos\beta_0 + \varkappa r'\cos\beta'_0\cos(\lambda'_0-\lambda_0),\\
\rho\cos\beta\sin(\lambda-\lambda_0) &= \varkappa r'\cos\beta'_0\sin(\lambda'_0-\lambda_0).
\end{aligned}$$

In the developments in infinite series which follow, the eccentricities of the orbits will be regarded as small quantities of the first order, the squares of the inclinations of the orbits on the plane of maximum areas as quantities of the third order, and $\varkappa$ also as a quantity of the third order. Then all terms, whose order is higher than the sixth, will be neglected. This degree of approximation will be found amply sufficient for the most refined investigations.

Under these conditions, we get

$$\begin{aligned}
\log\rho &= \log r + \tfrac{1}{2}\log\left[1 + 2\varkappa\frac{r'}{r}s + \varkappa^2\frac{r'^2}{r^2}\right]\\
&= \log r + \varkappa\frac{r'}{r}s + \tfrac{1}{2}\varkappa^2\frac{r'^2}{r^2}(1+2s^2),\\
\lambda &= \lambda_0 + \varkappa\frac{r'\cos\beta'_0}{r\cos\beta_0}\sin(\lambda'_0-\lambda_0) - \tfrac{1}{2}\varkappa^2\frac{r'^2}{r^2}\sin 2(\lambda'-\lambda_0),\\
\beta &= \beta_0 + \varkappa\frac{r'}{r}\beta'_0 - \varkappa\frac{r'}{r}s\beta_0.
\end{aligned}$$

We will write η for $\sin^2\frac{1}{2}i$. Then, to the sufficient degree of approximation,

$$\varkappa\frac{r'}{r}s = -\varkappa\frac{r'}{r}\cos(v-v'+g-g') + 2\varkappa(\eta+\eta')^2\frac{a'}{a}\sin(l+g)\sin(l'+g').$$

In like manner

$$\begin{aligned}
\varkappa\frac{r'\cos\beta'_0}{r\cos\beta_0}\sin(\lambda'_0-\lambda_0) &= \varkappa(1+\eta^2-\eta'^2)\frac{r'}{r}\sin(v-v'+g-g')\\
&\quad -\varkappa\eta^2\frac{a'}{a}\sin(3l-l'+3g-g') + \varkappa\eta'^2\frac{a'}{a}\sin(l+l'+g+g').
\end{aligned}$$

The expressions for λ_0 and β_0 in terms of elliptic elements are given by Delaunay.* Log r, as well as the following expressions

$$\frac{r'}{a'}{}^{\cos}_{\sin}(v'+g') = -\tfrac{3}{2}e'{}^{\cos}_{\sin}g'+[1-\tfrac{1}{2}e'^2]{}^{\cos}_{\sin}(l'+g')+[\tfrac{1}{2}e'-\tfrac{3}{8}e'^3]{}^{\cos}_{\sin}(2l'+g')$$
$$+\tfrac{3}{8}e'^2{}^{\cos}_{\sin}(3l'+g')+\tfrac{1}{3}e'^3{}^{\cos}_{\sin}(4l'+g')$$
$$\pm\tfrac{1}{8}e'^2{}^{\cos}_{\sin}(l'-g')\pm\tfrac{1}{24}e'^3{}^{\cos}_{\sin}(2l'-g'),$$

$$\frac{a}{r}{}^{\cos}_{\sin}(v+g) = -[\tfrac{1}{2}e+\tfrac{1}{8}e^3]{}^{\cos}_{\sin}g+[1-e^2]{}^{\cos}_{\sin}(l+g)+[\tfrac{3}{2}e-\tfrac{7}{4}e^3]{}^{\cos}_{\sin}(2l+g)$$
$$+\tfrac{17}{8}e^2{}^{\cos}_{\sin}(3l+g)+\tfrac{71}{24}e^3{}^{\cos}_{\sin}(4l+g)$$
$$\mp\tfrac{1}{8}e^2{}^{\cos}_{\sin}(l-g)\mp\tfrac{1}{12}e^3{}^{\cos}_{\sin}(2l-g),$$

are found in a memoir by Prof. Cayley.† With these data we get

$$\begin{aligned}
\log\rho = \log a &+ \tfrac{1}{4}e^2 + \tfrac{1}{32}e^4 + \tfrac{1}{96}e^6 + \varkappa^2\frac{a'^2}{a^2} \\
&-[e-\tfrac{3}{8}e^3-\tfrac{1}{64}e^5]\cos l-[\tfrac{3}{4}e^2-\tfrac{11}{24}e^4+\tfrac{3}{64}e^6]\cos 2l \\
&-[\tfrac{17}{24}e^3-\tfrac{77}{128}e^5]\cos 3l-[\tfrac{71}{96}e^4-\tfrac{129}{160}e^6]\cos 4l \\
&-\tfrac{523}{640}e^5\cos 5l-\tfrac{899}{960}e^6\cos 6l \\
&-\varkappa\frac{a}{a}\Big\{[1-e^2-\tfrac{1}{2}e'^2-(\eta+\eta')^2]\cos(l-l'+g-g') \\
&\quad+[\tfrac{3}{2}e-\tfrac{7}{4}e^3-\tfrac{3}{4}ee'^2]\cos(2l-l'+g-g')+[-\tfrac{3}{2}e'+\tfrac{3}{2}e^2e'] \\
&\qquad\qquad\times\cos(l+g-g') \\
&\quad+[-\tfrac{1}{2}e-\tfrac{1}{8}e^3+\tfrac{1}{4}ee'^2]\cos(l'-g+g')+[\tfrac{1}{2}e'-\tfrac{3}{8}e'^3-\tfrac{1}{2}e^2e'] \\
&\qquad\qquad\times\cos(l-2l'+g-g') \\
&\quad+\tfrac{17}{8}e^2\cos(3l-l'+g-g')-\tfrac{1}{8}e^2\cos(l+l'-g+g') \\
&\quad+\tfrac{3}{4}ee'\cos(g-g')-\tfrac{9}{4}ee'\cos(2l+g-g') \\
&\quad-\tfrac{1}{4}ee'\cos(2l'-g+g')+\tfrac{3}{4}ee'\cos(2l-2l'+g-g') \\
&\quad+\tfrac{3}{8}e'^2\cos(l-3l'+g-g')+\tfrac{1}{8}e'^2\cos(l+l'+g-g') \\
&\quad+\tfrac{71}{24}e^3\cos(4l-l'+g-g')-\tfrac{1}{12}e^3\cos(2l+l'-g+g') \\
&\quad-\tfrac{51}{16}e^2e'\cos(3l+g-g')+\tfrac{3}{16}e^2e'\cos(l-g+g') \\
&\quad+\tfrac{17}{16}e^2e'\cos(3l-2l'+g-g')-\tfrac{1}{16}e^2e'\cos(l+2l'-g+g') \\
&\quad-\tfrac{3}{16}ee'^2\cos(3l'-g+g')+\tfrac{9}{16}ee'^2\cos(2l-3l'+g-g') \\
&\quad-\tfrac{1}{16}ee'^2\cos(l'+g-g')+\tfrac{3}{16}ee'^2\cos(2l+l'+g-g') \\
&\quad+\tfrac{1}{3}e'^3\cos(l-4l'+g-g')+\tfrac{1}{24}e'^3\cos(l+2l'+g-g')
\end{aligned}$$

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* *Théorie du Mouvement de la Lune.* Tom. I. pp. 56–59.

† *Tables of the Developments of Functions in the Theory of Elliptic Motion. Mem. Roy. Astr. Soc.*, Vol. XXIX, p. 191.

$$+(\eta+\eta')^2\cos(l+l'+g+g')\Big\}$$

$$+\tfrac{1}{2}\varkappa^2\frac{a'^2}{a^2}\cos(2l-2l'+2g-2g'),$$

$$\lambda = l+g+h+[2e-\tfrac{1}{4}e^3+\tfrac{5}{96}e^5]\sin l+[\tfrac{5}{4}e^2-\tfrac{11}{24}e^4+\tfrac{17}{192}e^6]\sin 2l$$
$$+[\tfrac{13}{12}e^3-\tfrac{43}{64}e^5]\sin 3l+[\tfrac{103}{96}e^4-\tfrac{451}{480}e^6]\sin 4l$$
$$+\tfrac{1097}{960}e^5\sin 5l+\tfrac{1223}{960}e^6\sin 6l$$
$$+[-\eta^2-\eta^4+4\eta^2e^2]\sin(2l+2g)+\tfrac{1}{2}\eta^4\sin(4l+4g)$$
$$+[-2\eta^2e+\tfrac{27}{4}\eta^2e^3]\sin(3l+2g)+[2\eta^2e-\tfrac{7}{4}\eta^2e^3]\sin(l+2g)$$
$$-\tfrac{13}{4}\eta^2e^2\sin(4l+2g)-\tfrac{3}{4}\eta^2e^2\sin 2g$$
$$-\tfrac{59}{12}\eta^2e^3\sin(5l+2g)+\tfrac{1}{12}\eta^2e^3\sin(l-2g)$$
$$+\varkappa\frac{a'}{a}\Big\{[1-e^2-\tfrac{1}{2}e'^2+\eta^2-\eta'^2]\sin(l-l'+g-g')$$
$$+[\tfrac{3}{2}e-\tfrac{7}{4}e^3-\tfrac{3}{4}ee'^2]\sin(2l-l'+g-g')+[\tfrac{1}{2}e+\tfrac{1}{8}e^3-\tfrac{1}{4}ee'^2]$$
$$\times\sin(l'-g+g')$$
$$+[-\tfrac{3}{2}e'+\tfrac{3}{2}e^2e']\sin(l+g-g')+[\tfrac{1}{2}e'-\tfrac{3}{8}e'^3-\tfrac{1}{2}e^2e']$$
$$\times\sin(l-2l'+g-g')$$
$$+\tfrac{17}{8}e^2\sin(3l-l'+g-g')+\tfrac{1}{8}e^2\sin(l+l'-g+g')$$
$$+\tfrac{3}{4}ee'\sin(g-g')-\tfrac{9}{4}ee'\sin(2l+g-g')+\tfrac{1}{4}ee'\sin(2l'-g+g')$$
$$+\tfrac{3}{4}ee'\sin(2l-2l'+g-g')+\tfrac{3}{8}e'^2\sin(l-3l'+g-g')$$
$$+\tfrac{1}{8}e'^2\sin(l+l'+g-g')$$
$$+\tfrac{71}{24}e^3\sin(4l-l'+g-g')+\tfrac{1}{12}e^3\sin(2l+l'-g+g')$$
$$-\tfrac{51}{16}e^2e'\sin(3l+g-g')-\tfrac{3}{16}e^2e'\sin(l-g+g')$$
$$+\tfrac{17}{16}e^2e'\sin(3l-2l'+g-g')+\tfrac{1}{16}e^2e'\sin(l+2l'-g+g')$$
$$+\tfrac{3}{16}ee'^2\sin(3l-g+g')+\tfrac{9}{16}ee'^2\sin(2l-3l'+g-g')$$
$$-\tfrac{1}{16}ee'^2\sin(l'+g-g')+\tfrac{3}{16}ee'^2\sin(2l+l'+g-g')$$
$$+\tfrac{1}{3}e'^3\sin(l-4l'+g-g')+\tfrac{1}{24}e'^3\sin(l+2l'+g-g')$$
$$-\eta^2\sin(3l-l'+3g-g')+\eta'^2\sin(l+l'+g+g')\Big\}$$
$$+\tfrac{1}{2}\varkappa^2\frac{a'^2}{a^2}\sin(2l-2l'+2g-2g'),$$

$$\beta = [2\eta-2\eta e^2+\tfrac{7}{32}\eta e^4]\sin(l+g)-\tfrac{1}{3}\eta^3\sin(3l+3g)$$
$$+[2\eta e-\tfrac{5}{2}\eta e^3]\sin(2l+g)-2\eta e\sin g$$
$$+[\tfrac{9}{4}\eta e^2-\tfrac{27}{8}\eta e^4]\sin(3l+g)+[\tfrac{1}{4}\eta e^2-\tfrac{1}{24}\eta e^4]\sin(l-g)$$
$$+\tfrac{8}{3}\eta e^3\sin(4l+g)+\tfrac{1}{6}\eta e^3\sin(2l-g)$$
$$+\tfrac{625}{192}\eta e^4\sin(5l+g)+\tfrac{9}{64}\eta e^4\sin(3l-g)$$
$$-\eta^3e\sin(4l+3g)+\eta^3e\sin(2l+3g)$$
$$+\varkappa\frac{a'}{a}\Big\{\eta\sin(2l-l'+2g-g')+[\eta+2\eta']\sin(l'+g')$$
$$+\tfrac{5}{2}\eta e\sin(3l-l'+2g-g')-\tfrac{3}{2}\eta e\sin(l-l'+2g-g')$$

$$
\left.\begin{aligned}
&-\tfrac{3}{2}\eta e' \sin(2l+2g-g')+\tfrac{1}{2}\eta e' \sin(2l-2l'+2g-g')\\
&+\tfrac{1}{2}[\eta+2\eta']\, e \sin(l+l'+g')-\tfrac{1}{2}[\eta+2\eta']\, e \sin(l-l'-g')\\
&+\tfrac{1}{2}[\eta+2\eta']\, e' \sin(2l'+g')-\tfrac{3}{2}[\eta+2\eta'] e' \sin g'
\end{aligned}\right\}.
$$

As written, these expressions give the coordinates of Jupiter. Those of Saturn are obtained by removing the accent from all the accented symbols, and applying it to those which are unaccented, $\varkappa$ excepted, for which we have $\varkappa' = \varkappa$. Also it is to be remembered that we have $h' = h + 180°$.

The coordinates of the two planets are obtained by employing in these formulæ, for the quantities involved in them, the values they actually have at the time in question. The latter are determined by the differential equations previously given; but instead of integrating these equations in one step, we may, as Delaunay has done in the lunar theory, divide the process into a series of transformations of the variables involved; each of which must be made not only in the expressions for $\log \rho$, λ, β, $\log \rho'$, λ', β', but also in R.

As the introduction of l as the independent variable does not appear to be advantageous, we will suppose that the six variables L, L', Γ, l, l', γ are employed and that t is the independent variable.

Delaunay's method, somewhat amplified, amounts to this:— selecting the argument $\theta = il + i'l' + i''\gamma$, suppose, for the moment, that R is limited to the terms

$$-B - A_1 \cos(il+i'l'+i''\gamma) - A_2 \cos 2(il+i'l'+i''\gamma) + \ldots,$$

where B, A_1 &c., are functions of L, L' and Γ only. Then if it is found that the differential equations, corresponding to this limited R, are satisfied by the infinite series

$$
\begin{aligned}
\theta &= \theta_0(t+c) + \theta_1 \sin \theta_0(t+c) + \theta_2 \sin 2\theta_0(t+c) + \ldots,\\
l &= (l) + l_0(t+c) + l_1 \sin \theta_0(t+c) + l_2 \sin 2\theta_0(t+c) + \ldots,\\
l' &= (l') + l'_0(t+c) + l'_1 \sin \theta_0(t+c) + l'_2 \sin 2\theta_0(t+c) + \ldots,\\
\gamma &= (\gamma) + \gamma_0(t+c) + \gamma_1 \sin \theta_0(t+c) + \gamma_2 \sin 2\theta_0(t+c) + \ldots,\\
L &= L_0 + L_1 \cos \theta_0(t+c) + L_2 \cos 2\theta_0(t+c) + \ldots,\\
L' &= L'_0 + L'_1 \cos \theta_0(t+c) + L'_2 \cos 2\theta_0(t+c) + \ldots,\\
\Gamma &= \Gamma_0 + \Gamma_1 \cos \theta_0(t+c) + \Gamma_2 \cos 2\theta_0(t+c) + \ldots,
\end{aligned}
$$

where c, (l), (l') and (γ) are arbitrary constants, the last three being equivalent to two independent constants, as we have the relation

$$i(l) + i'(l') + i''(\gamma) = 0,$$

and all the other coefficients are known functions of three other constants a, a' and e, we can replace

$$
\begin{aligned}
&L \text{ by } L_0 + L_1 \cos(il+i'l'+i''\gamma) + L_2 \cos 2(il+i'l'+i''\gamma) + \ldots,\\
&L' \text{ by } L'_0 + L'_1 \cos(il+i'l'+i''\gamma) + L'_2 \cos 2(il+i'l'+i''\gamma) + \ldots,
\end{aligned}
$$

$$\begin{aligned}
&\Gamma \text{ by } \Gamma_0 + \Gamma_1 \cos(il+i'l'+i''\gamma) + \Gamma_2 \cos 2(il+i'l'+i''\gamma) + \ldots,\\
&l \text{ by } l + l_1 \sin(il+i'l'+i''\gamma) + l_2 \sin 2(il+i'l'+i''\gamma) + \ldots,\\
&l' \text{ by } l' + l'_1 \sin(il+i'l'+i''\gamma) + l'_2 \sin 2(il+i'l'+i''\gamma) + \ldots,\\
&\gamma \text{ by } \gamma + \gamma_1 \sin(il+i'l'+i''\gamma) + \gamma_2 \sin 2(il+i'l'+i''\gamma) + \ldots,
\end{aligned}$$

and will have, for determining the new variables l, l', γ, a, a', e, precisely the same differential equations as we started with, provided we make all these substitutions in the function R, and regard the new variables L, L', Γ as connected with a, a', e by the relations

$$\begin{aligned}
L &= L_0 + \tfrac{1}{2}(\theta_1 L_1 + 2\theta_2 L_2 + \ldots),\\
L' &= L'_0 + \tfrac{1}{2}(\theta_1 L'_1 + 2\theta_2 L'_2 + \ldots),\\
\Gamma &= \Gamma_0 + \tfrac{1}{2}(\theta_1 \Gamma_1 + 2\theta_2 \Gamma_2 + \ldots).
\end{aligned}$$

It will be perceived that as long as we are dealing with terms of R, whose arguments involve l or l' or both, the second members of the three equations, last written, have values which differ from the elliptic values of L, L' and Γ only by quantities of the second order with respect to disturbing forces. Hence, if we propose to neglect third order terms, until we have reduced R to a function of the argum't γ only, we can assume that L, L' and Γ which are the elements conjugate to the arguments l, l' and γ, are expressed throughout in terms of a, a' and e, in the same way as in the elliptic theory. It may be added that these third order terms are found in experience to be much smaller than those which arise in other ways.

[To be continued.]

SOME RELATIONS DEDUCED FROM EULER'S THEOREM ON THE CURVATURE OF SURFACES.

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LET α = angle of any normal section with line of maximum curvature,
k_α = curvature of this normal section,
k_m = maximum curvature,
k_n = minimum curvature; then by Euler's theorem :

$$k_\alpha = k_m \cos^2\alpha + k_n \sin^2\alpha, \tag{1}$$
$$\therefore\ k_{\alpha+\frac{1}{2}\pi} = k_m \sin^2\alpha + k_n \cos^2\alpha.$$

Adding, we obtain the well known relation:

$$k_\alpha + k_{\alpha+\frac{1}{2}\pi} = k_m + k_n = \text{constant}. \tag{2}$$

But multiplying we have

$$\begin{aligned}
k_\alpha k_{\alpha+\frac{1}{2}\pi} &= (k_m^2 + k_n^2)\sin^2\alpha\cos^2\alpha + k_m k_n(\cos^4\alpha + \sin^4\alpha)\\
&= \tfrac{1}{4}(k_m + k_n)^2 \sin^2 2\alpha + k_m k_n \cos^2 2\alpha. \qquad (3)
\end{aligned}$$

Place $\quad k_{m+n} = \frac{1}{2}(k_m + k_n) = \frac{1}{2}(k_\alpha + k_{\alpha+\frac{1}{2}\pi}) = k_{\frac{1}{4}\pi}, \qquad (4)$